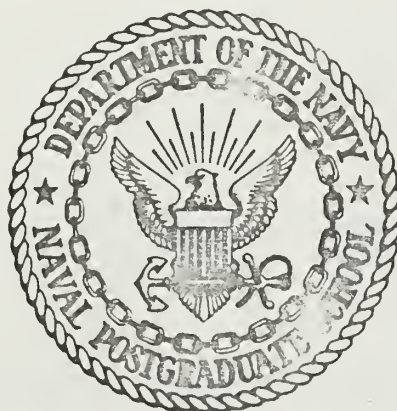


SUBSEMIGROUP STRUCTURE OF FINITE
TRANSFORMATION SEMIGROUPS

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THESIS

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by

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ABSTRACT

Both necessary conditions and sufficient conditions in order that a subset of a finite transformation semigroup be a subsemigroup are developed in this paper. The existence of several subsemigroups of various orders is established. Also, some results concerning idempotents and generators of idempotents are proved. Then certain classes of subsemigroups are defined according to their idempotent structure and isomorphisms are demonstrated between these classes. Examples from the transformation semigroup on three elements are supplied throughout the paper.

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I. INTRODUCTION

In this paper the algebraic structure of the full transformation semigroup over a finite domain is explored. For notation let S_n be an arbitrary set with n elements and put $T_n = \left\{ \text{all functions } \rho \mid \rho: S_n \rightarrow S_n \right\}$. Then T_n is a semigroup under the binary operation of functional composition and is called the full transformation semigroup on n elements. The order of T_n is n^n . Without loss of generality S_n can be taken to be $\{1, 2, \dots, n\}$, and an arbitrary element of ρ of T_n may be considered as an order n -tuple $(\alpha_1, \dots, \alpha_n)$, where $\alpha_j = \rho(j)$ ¹.

The problem that motivated this paper is to determine the number of subsemigroups of T_n for orders one through n^n . The solution of this problem for T_2 is contained in Section III. These results may be obtained simply by checking all possible subsets for closure (note that associativity automatically carries over to any subset of T_n). However, in general, theoretical tools are essential, because the number of subsets of T_n that are candidates for subsemigroups is $2^{(n^n)}$. Even for T_3 , this number is 134,217,728. Thus in order to write a program with which the subsemigroups of T_n could feasibly be computed, criteria simpler than closure are needed. The search for such criteria led to the theoretical results contained herein. Incorporation of Theorem 7 into the program reduced its run time sufficiently to allow the computation of the subsemigroups of T_3 for orders one through ten and nineteen through twenty-seven. The material following Theorem 7 in Section V is useful in

¹Orval Sweeney, LT(jg), USN, Numerical Properties of the Full Transformation Semigroup on a Finite Domain, p. 6, 7, M.S. Thesis, NPS, 1968.

determining the information needed to apply the theorem to any specific T_n . The results in Section VI were motivated by an attempt to reduce run time by eliminating the need for checking certain classes of subsets of T_n . Although application of these results failed to accomplish the original goal (because the time required to apply this check to all subsets is greater than the time saved by eliminating the checking of certain classes of subsets for closure), the results are presented for their mathematical interest.

II. ISOMORPHISMS

Included in the algebraic structure of the full transformation semigroup, discussed in this paper, are the isomorphisms that exist between certain semigroups of various orders. The usual concept of isomorphism between two semigroups is as follows.

Two semigroups, S_1 and S_2 , are isomorphic if there exists a bijection $\eta: S_1 \rightarrow S_2$ such that $\eta(x) \cdot \eta(y) = \eta(xy)$, for every x, y in S_1 . Under this concept of isomorphism, for each given n , there is a fixed number of non-isomorphic semigroups of order n . For example:

<u>n</u>	<u>Number of Semigroups Distinct up to an Isomorphism</u> ²
1	1
2	5
3	24
4	188

Also, under this concept of isomorphism, any semigroup of order n may be isomorphically imbedded into T_m , for $m \geq n + 1$ ³. For example, in T_3 , all five abstract semigroups of order 2 are realized as subsemigroups of T_3 .

However, there is a more general concept of isomorphism, which seems to be more applicable to transformation semigroups. Here this is called a τ -isomorphism.

²James Cullen, LT(jg), USN, An Algorithm for Computing Non-Isomorphic Semigroups of Finite Order, p. 15, M.S. Thesis, NPS, 1968.

³Sweeney, p. 10.

Definition:⁴ Let S, T be subsemigroups of the transformation semigroup over the sets X and Y respectively. Then S and T are τ -isomorphic if there exists a pair of bijections (φ, η) $\varphi: X \rightarrow Y$ and $\eta: S \rightarrow T$ such that $[\eta(s)]\varphi(x) = \varphi(s(x))$, for every s in S and all x in X .

Denote that S and T are τ -isomorphic by $S \cong T$.

It is easy to show that if $S \cong T$, then S is also isomorphic to T , under the usual concept of isomorphism, defined earlier. However, the converse is not true, even if the underlying sets are the same (i.e., if $X = Y$). For an example of this see page 4 of the paper by C. Wilde and T. Jayachandran.

Lemma 1: Let σ and σ' be transformation semigroups defined over the set S_n . Then $\sigma \cong \sigma'$ if and only if there exists a bijection $\varphi: S_n \rightarrow S_n$ and a surjection $\eta: \sigma \rightarrow \sigma'$; $\rho \mapsto \varphi\rho\varphi^{-1}$.

Proof: Assume $\sigma \cong \sigma'$. Thus there exist a pair of bijections (φ, η) with $\varphi: S_n \rightarrow S_n$ and $\eta: \sigma \rightarrow \sigma'$, such that $[\eta(\rho)]\varphi(i) = \varphi(\rho(i))$, for all $\rho \in \sigma$ and all $i \in S_n$. Therefore $\varphi^{-1}\eta(\rho)\varphi(i) = \rho(i)$, for all $i \in S_n$. Hence $\varphi^{-1}\eta(\rho)\varphi = \rho$ or $\eta(\rho) = \varphi\rho\varphi^{-1}$, for all ρ in σ . Conversely, assume there exists a bijection $\varphi: S_n \rightarrow S_n$ and a surjection $\eta: \sigma \rightarrow \sigma'$; $\rho \mapsto \varphi\rho\varphi^{-1}$. This implies $[\eta(\rho)]\varphi(i) = [\varphi\rho\varphi^{-1}](i) = \varphi[\rho(i)]$, for all $i \in S_n$ and all $\rho \in \sigma$. Now, we show η is a bijection. Let $\rho_1, \rho_2 \in \sigma$ such that $\rho_1 \neq \rho_2$. This implies there exists an i in S_n such that $\rho_1(i) \neq \rho_2(i)$. We note that $\varphi(i) = j$ and hence $j = \varphi^{-1}(i)$.

$$\eta(\rho_1)(j) = \varphi\rho_1\varphi^{-1}(j) = \varphi\rho_1(i) \neq \varphi\rho_2(i) = \varphi\rho_2[\varphi^{-1}(j)] = \eta(\rho_2)(j)$$

Therefore $\eta(\rho_1) \neq \eta(\rho_2)$. Thus, η is injective and hence a bijection.

This implies $\sigma \cong \sigma'$. \blacktriangle

⁴Carroll Wilde and Toke Jayachandran, Amenable Transformation Semigroups, p. 4, J. Australian Math. Soc., to appear (1970).

Lemma 1 allows us to set up a simple procedure for determining whether two subsemigroups of T_n are τ -isomorphic. For notation, let G_n represent the symmetric group on n elements, i.e. the collection of bijections from S_n into itself. $|G_n| = n!$ The following Corollary is an immediate consequence of Lemma 1.

Corollary 1: Let σ, σ' be subsemigroups of T_n . Then $\sigma \approx \sigma'$ if and only if $\sigma' = \{\varphi\sigma\varphi^{-1} : \varphi \in G_n\}$ for some $\varphi \in G_n$.

III. ANALYSIS OF T_2

Notation: If $\rho_1, \rho_2 \in T_n$, then $(\rho_1 \rho_2)(i) = \rho_1[\rho_2(i)]$.

We have $S_2 = \{1, 2\}$ and $T_2 = \{1, 2, 3, 4\}$, where

$$1 = (1, 2)$$

$$2 = (1, 1)$$

$$3 = (2, 2)$$

$$4 = (2, 1)$$

The use of integers in two different capacities should not be confusing, since the meaning will be clear from the context. The multiplication table for T_2 is

.	1	2	3	4
1	1	2	3	4
2	2	2	2	2
3	3	3	3	3
4	4	3	2	1

The subsemigroup structure of T_2 is:

<u>Order</u>	<u>Number of Subsemigroups</u>	<u>List of Subsemigroups</u>
1	3	$\{1\}$, $\{2\}$, $\{3\}$
2	4	$\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$
3	1	$\{1,2,3\}$
4	1	$\{1,2,3,4\}$

Order 1: All three subsemigroups of order 1 are isomorphic, as there is only one abstract semigroup of order 1.

Order 2: $\{1,2\}$ and $\{1,3\}$ are the only two subsemigroups of order 2 that are isomorphic. Thus we see that in T_2 only three of the

possible five distinct semigroups on two elements are realized as subsemigroups of T_2 . $\{1,2\}$ and $\{1,3\}$ are also τ -isomorphic.

Order 3: $\{1,2,3\}$ is the subsemigroup of order 3 from T_2 that is predicted by Corollary 5.

Order 4: This is the full transformation semigroup itself, which has order n^n .

This type of analysis, which is simple for T_2 , becomes very involved for T_n , where $n \geq 3$. We will first of all, derive some sufficient conditions, that a given subset of T_n be a subsemigroup. These will predict a certain number of subsemigroups of various orders. However, these predictions do not nearly give a complete listing of all the subsemigroups of T_n . Later in the paper we will develop some necessary conditions, which make the search for subsemigroups easier, but which do not entirely eliminate the need for an exhaustive search.

At this time, it is convenient to list the elements of T_3 as they will be referred to several times. Let $S_3 = \{1,2,3\}$ and $T_3 = \{1,2,3,\dots,27\}$,

1 = (111)	10 = (211)	19 = (311)
2 = (112)	11 = (212)	20 = (312)
3 = (113)	12 = (213)	21 = (313)
4 = (121)	13 = (221)	22 = (321)
5 = (122)	14 = (222)	23 = (322)
6 = (123)	15 = (223)	24 = (323)
7 = (131)	16 = (231)	25 = (331)
8 = (132)	17 = (232)	26 = (332)
9 = (133)	18 = (233)	27 = (333)

IV. SOME MISCELLANEOUS SUBSEMIGROUPS

The collection of transformations in T_n , under which a given set, $A \subseteq S_n$ is invariant, forms a subsemigroup.

Theorem 1: If $A \subseteq S_n$, then $P_A \equiv \{\rho \in T_n : \rho(A) = A\}$ is a subsemigroup of T_n . Furthermore, if $|A| = k$, then $|P_A| = k!n^{(n-k)}$, for $k = 1, \dots, n$.

Proof: Let $|A| = k$ and ρ be in P_A . Then k of the positions of ρ must be assigned one of the k different values from A . There are $k!$ ways to do this. The other $(n-k)$ positions may be assigned any one of n values. Hence for each permutation of the k elements, there are $n^{(n-k)}$ possibilities. Thus $|P_A| = k!n^{(n-k)}$. Now, let ρ_1, ρ_2 be in P_A . $(\rho_1\rho_2)(A) = \rho_1[\rho_2(A)] = \rho_1(A) = A$. Hence $(\rho_1\rho_2)$ is in P_A and P_A is a subsemigroup. Δ

Corollary 2: In T_n , there are at least $\binom{n}{k}$ subsemigroups of order $k!n^{(n-k)}$, $k = 1, \dots, n$.

Proof: There are $\binom{n}{k}$ ways to choose a set A of order k from S_n . Thus there are $\binom{n}{k}$ different subsemigroups, P_A , each having order $k!n^{(n-k)}$. Δ

Remark: (i) If $|A| = n$, then the single P_A obtained is the symmetric group on n elements, G_n .

(ii) If $|A| = 1$, the n subsemigroups predicted are the n subsemigroups of order 1, each of which corresponds to a constant transformation.

Theorem 2: If $A, B \subseteq S_n$ such that $|A| = |B|$, then $P_A \cong P_B$.

Proof: Define a bijection $\varphi: S_n \rightarrow S_n$, such that $\varphi(A) = B$. Also define a mapping $\eta: P_A \rightarrow P_B$; $\rho \mapsto \varphi\rho\varphi^{-1}$. We must show that η is a surjection of P_A onto P_B .

$[\eta(\rho)](B) = \varphi\varphi^{-1}(B) = \varphi(\rho(A)) = \varphi(A) = B$. Therefore $\eta(\rho)$ is in P_B for every ρ in P_A . Let $s \in P_B$. We must show there exists a ρ in P_A such that $\eta(\rho) = s$. Consider the transformation ρ defined by $\rho[\varphi^{-1}(i)] = \varphi^{-1}[s(i)]$ for all i in S_n . This implies $\varphi\rho\varphi^{-1}(i) = s(i)$ or $\eta(\rho) = s$, but is $\rho \in P_A$? $\rho(A) = \rho[\varphi^{-1}(B)] = \varphi^{-1}[s(B)] = \varphi^{-1}(B) = A$. Therefore $\rho \in P_A$ and hence η is a surjection. Finally, by Lemma 1, we conclude $P_A \cong P_B$. \blacktriangle

The collection of transformations in T_n that leave certain given elements of S_n unchanged also forms a subsemigroup.

Theorem 3: If $A \subseteq S_n$, then $K_A \equiv \{\rho \in T_n : \rho(i) = i, \text{ for all } i \in A\}$ is a subsemigroup of T_n . Furthermore, if $|A| = k$, then $|K_A| = n^{(n-k)}$, $k = 1, \dots, n$.

Proof: Let $|A| = k$ and ρ be in K_A . Then k of the elements in the range of ρ are fixed. The other $(n-k)$ elements may be any one of the n elements of S_n . Therefore $|K_A| = n^{(n-k)}$. Now let i be in A and $\rho_1, \rho_2 \in K_A$. Then $(\rho_1\rho_2)(i) = \rho_1[\rho_2(i)] = \rho_1(i) = i$. This implies $(\rho_1\rho_2) \in K_A$ and hence K_A is a subsemigroup. \blacktriangle

Corollary 3: In T_n , there are at least $\binom{n}{k}$ subsemigroups of order $n^{(n-k)}$, $k = 1, \dots, n$.

Proof: There are $\binom{n}{k}$ ways to choose a set A of order k from S_n . Thus there are $\binom{n}{k}$ different subsemigroups, K_A , in T_n , each having order $n^{(n-k)}$. \blacktriangle

Remark: (i) For $k = 1$, this gives the same n subsemigroups as Corollary 2.

(ii) For $k = n$, this gives the one subsemigroup of order 1 that contains the identity transformation.

Theorem 4: If $A, B \subseteq S_n$ and $|A| = |B|$, then $K_A \cong K_B$.

Proof: The proof of this theorem exactly parallels that of Theorem 2.

We can also obtain a few more subsemigroups by modifying slightly the concept of invariance, employed in Theorem 1.

Theorem 5: If $A \subseteq S_n$, then $Q_A \equiv \left\{ \rho \in T_n : \rho(A) \subseteq A \right\}$ is a subsemigroup of T_n . Furthermore, if $|A| = k$ then $|Q_A| = k^k n^{(n-k)}$, $k = 1, \dots, n$.

Proof: Let $|A| = k$ and ρ be in Q_A . k of the positions of ρ may be assigned any one of the k values of A . There are k^k ways of accomplishing this. Then the $(n-k)$ other positions of ρ may be assigned any one of n values. Thus there are $k^k n^{(n-k)}$ different transformations in Q_A or $|Q_A| = k^k n^{(n-k)}$. Let $\rho_1, \rho_2 \in Q_A$. Then $(\rho_1 \rho_2)(A) = \rho_1[\rho_2(A)] = \rho_1(K)$, where $K \subseteq A$. But $\rho_1(K) \subseteq \rho_1(A) \subseteq A$. Hence $(\rho_1 \rho_2) \in Q_A$ and thus Q_A is a subsemigroup. \blacktriangle

Corollary 4: In T_n , there are at least $\binom{n}{k}$ subsemigroups of order $k^k n^{(n-k)}$, $k = 1, \dots, n$.

Proof: There are $\binom{n}{k}$ ways to choose a set A of order k from S_n . Here, however, the question arises that if $|A| = |B| = k$ and $A \neq B$, then is $Q_A \neq Q_B$? $A \neq B$ implies there exists an i in A such that i is not in B . If we let $\rho = (ii \dots i)$, then $\rho \in Q_A$ but $\rho \notin Q_B$. Hence $Q_A \neq Q_B$. Thus there are $\binom{n}{k}$ distinct subsemigroups, Q_A , of T_n , each having order $k^k n^{(n-k)}$. \blacktriangle

Remark: (i) For every $A \subseteq S_n$, if $\rho \in P_A$ then $\rho \in Q_A$. Hence $P_A \subseteq Q_A$.

(ii) For $k = 1$, the n subsemigroups predicted here are the same as in both Corollary 2 and 3.

(iii) For $k = n$, the subsemigroup predicted is of order n^n and hence is T_n itself.

Theorem 6: If $A, B \subseteq S_n$ and $|A| = |B|$, then $Q_A \cong Q_B$.

Proof: The proof of this theorem exactly parallels that of Theorem 2.

As before, we let G_n represent the symmetric group on n elements.

Lemma 2: $(T_n - G_n)$ is a subsemigroup of order $n^n - n!$.

Proof: First of all, let $\rho, \eta \in (T_n - G_n)$ and then show that $(\rho\eta) \notin G_n$. $\eta \notin G_n$ implies there exists $i \in S_n$ such that $\eta(j) \neq i$ for every j in S_n . Hence the order of the range of η is less than n . This implies the order of the range of $(\rho\eta)$ is less than n and therefore $(\rho\eta) \notin G_n$. Thus $(\rho\eta) \in (T_n - G_n)$ and $(T_n - G_n)$ is a subsemigroup. Finally $|G_n| = n!$, so $|T_n - G_n| = n^n - n!$. \blacktriangle

Remark: $(T_3 - G_3)$ is the only subsemigroup of T_3 of order 21.

Notation: Let \downarrow_n represent the identity transformation of T_n .

Lemma 3: If X_n is a subsemigroup of T_n of order x such that $\downarrow_n \notin X_n$, then $(X_n \cup \downarrow_n)$ is a subsemigroup of order $(x+1)$.

Proof: Let ρ be an arbitrary element of X_n . Then, $\rho\downarrow_n = \downarrow_n\rho = \rho \in X_n$. Hence $(X_n \cup \downarrow_n)$ is a subsemigroup. \blacktriangle

Corollary 5: $(T_n - G_n + \downarrow_n)$ is a subsemigroup of order $(n^n - n! + 1)$.

Proof: $\downarrow_n \in G_n$ and thus $\downarrow_n \notin T_n - G_n$. Therefore we apply Lemma 2 and Lemma 3 to obtain the result. \blacktriangle

Remark: $(T_3 - G_3 + \downarrow_3)$ is the only subsemigroup of T_3 of order 22.

Lemma 4: Let A_n be the alternating normal subgroup of G_n . Then $T_n - (G_n - A_n)$ is a subsemigroup of T_n of order $(n^n - \frac{n!}{2})$.

Proof: Let $\eta, \rho \in T_n - (G_n - A_n) \equiv X$. We consider two cases.

Case 1: Here both η and ρ are in A . This implies $(\eta\rho)$ is in A_n and hence $\eta\rho$ is in X .

Case 2: Here at least one of the transformations ρ, η is not in A_n . Without loss of generality suppose $\eta \notin A_n$. This implies $\eta \notin G_n$ and hence $(\eta\rho) \notin G_n$. Thus $\eta\rho$ is in X . Therefore in either case X is a subsemigroup. From group theory $|A_n| = \frac{n!}{2}$ and consequently $|X| = n^n - \frac{n!}{2}$. \blacktriangle

Remark: Once again we note that $T_3 - (G_3 - A_3)$ is the only subsemigroup of T_3 of order 24.

V. IDEMPOTENTS OF T_n

In this section the idempotents of T_n are studied. This investigation will lead to some necessary conditions for subsemigroups of various orders.

It is known that if a semigroup has an element, say s , of finite order, then it has an idempotent. The process for proving that this idempotent exists, is to find two integers, $a < b$, such that $s^a = s^b$. Then there exists an integer n such that $s^{n(b-a)}$ is idempotent. The smallest n that will suffice in every case is $n = a^5$.

Definition: The idempotent, e , obtained from the element s by the above process is called the "idempotent generated by s ". We denote this by $s \text{ gen } e$.

Note: In a finite semigroup, every element is of finite order.

Lemma 5: Let $s \in S$, a finite semigroup. Then $s \text{ gen } e$ if and only if $s^k \text{ gen } e$, for all integers k .

Proof: First of all, assume $s \text{ gen } e$. Let $a < b$ be integers such that $s^a = s^b$. Then $s^{ka} = s^{kb}$ or $(s^k)^a = (s^k)^b$. Thus $s^k \text{ gen } (s^k)^{a(b-a)} = [s^{a(b-a)}]^k = e^k = e$, so that $s^k \text{ gen } e$. Conversely assume $s^k \text{ gen } e$. This implies there exist $a < b$ such that $(s^k)^a = (s^k)^b$ and $e = (s^k)^{a(b-a)} = s^{ka(b-a)}$. If we note that $s^{ka} = s^{kb}$, then $s \text{ gen } s^{ka(kb-ka)}$. However $s^{ka(kb-ka)} = [s^{ka(b-a)}]^k = e^k = e$. Therefore $s \text{ gen } e$. \blacktriangle

Corollary 6: Let $s \in S$, a finite semigroup. Then $s \text{ gen } e$ if and only if $s^k = e$, for some integer k .

⁵H. S. Vandiver and Milo W. Weaver, Introduction to Arithmetic Factorization and Congruences from the Standpoint of Abstract Algebra, V. 65, No. 8, Part II, p. 48, October 1958.

Proof: First assume $s \text{ gen } e$. Thus $s^{a(b-a)} = e$, for some a and b , so that $k = a(b-a)$. Conversely assume $s^k = e$. Clearly every idempotent generates itself, so that $s^k \text{ gen } e$. Hence, applying Lemma 5, $s \text{ gen } e$. \blacktriangle

Lemma 6: Every element of a finite semigroup, say S , must generate one of the idempotents of S .

Proof: Assume there exists an $s \in S$ such that $s \text{ gen } e$, but $e \notin S$. Corollary 6 implies there exists an integer k such that $s^k = e$. Hence $s^k \in S$. This is a contradiction, since S is a semigroup. \blacktriangle

Notation: If e is an idempotent of T_n , we put

$$G_e \equiv \left\{ \rho \in T_n : \rho \text{ gen } e \right\}.$$

Now, let $\{e_1, \dots, e_k\}$ be the collection of idempotents of T_n ; where $|G_{e_1}| = \alpha_1, \dots, |G_{e_k}| = \alpha_k$. Since every element of T_n generates one and only one idempotent, it follows that $\sum_{i=1}^k \alpha_i = n^n$. Also every idempotent generates itself so that for all i , $\alpha_i \neq 0$. Order the collection $\{\alpha_i\}_{i=1, \dots, k}$ in decreasing order, $\{\alpha_1', \dots, \alpha_k'\}$.

Theorem 7: Any subsemigroup of T_n of order greater than $\sum_{j=1}^x \alpha_j'$ must contain at least $(x+1)$ idempotents, where $x = 1, \dots, (k-1)$.

Proof: Let σ be a subsemigroup of T_n of order p , where $p > \sum_{j=1}^x \alpha_j'$ and assume σ has less than $(x+1)$ idempotents. By Lemma 6, every element of σ must generate one of the idempotents of σ . Thus the most elements that σ can contain are all of the generators of all of its idempotents. Since $\{\alpha_j'\}_{j=1, \dots, k}$ are arranged in decreasing order, we conclude that the most elements σ can contain is $\sum_{j=1}^x \alpha_j'$. But we know $p < \sum_{j=1}^x \alpha_j'$, which is a contradiction. Hence σ has at least $(x+1)$ idempotents. \blacktriangle

As an example this theorem is now applied to T_3 . There are ten idempotents in T_3 and $\{\alpha_j'\}_{j=1, \dots, 10} = \{6, 3, 3, 3, 2, 2, 2, 2, 2, 2\}$.

Thus considering the subsemigroups of, say order 10, every one must have at least three idempotents. This concept is useful as a criterion that a subset of T_n be a subsemigroup when digitally computing subsemigroups of various orders.

Let us now look at idempotents and their generators in T_n for arbitrary n . The number of idempotents in T_n is known, but the number of generators a given idempotent has is, in general, still an open question. Thus we cannot give a closed expression for the collection $\{\alpha_j\}$, used above. However, we can draw some conclusions about these numbers.

Lemma : If $u \in T_n$, then u is an idempotent if and only if every point in the range of u is a fixed point of u .

Proof: See p. 15 of O. Sweeney's Thesis.

Now denote by I_k^n the collection of all idempotents in T_n that fix exactly k elements.

Lemma⁵ : $|I_k^n| = \binom{n}{k} k^{n-k}$, where $1 \leq k \leq n$.

Proof: By the preceding Lemma, the range of $u \in I_k^n$ has k elements. There are $\binom{n}{k}$ ways to choose the range, then each of the $(n-k)$ elements not in the range may be mapped by u to any one of the k range elements. Hence for every choice of a range set, there are k^{n-k} choices for the other elements. Thus there are $\binom{n}{k} k^{n-k}$ different idempotents which fix exactly k elements. Δ

Thus, since an idempotent can fix anywhere from one to n elements, the total number of idempotents in T_n is $\sum_{k=1}^n \binom{n}{k} k^{n-k}$.

Notation: Let $E \subseteq S_n$ such that $|E| = k$. Then $I_k^n(E) \equiv \left\{ \rho \in I_k^n : \rho(i) = i, \text{ for all } i \in E \right\}$.

Clearly $|I_k^n(E)| = k^{n-k}$ for every E of order k . Also note that $I_n^n(S_n) = \left\{ \text{id}_n \right\}$, where id_n is the identity transformation of T_n .

⁵Sweeney, p. 15

Notation: Let $E \subseteq S_n$ such that $|E| = k$. Put

$$G(E) \equiv \left\{ \rho \in T_n : \rho \text{ gen } e, \text{ for some } e \text{ in } I_k^n(E) \right\}.$$

Theorem 7: Let E_1, E_2 be arbitrary subsets of S_n , such that

$$|E_1| = |E_2| = k. \text{ Then } |G(E_1)| = |G(E_2)|.$$

Proof: Define a bijection φ such that $\varphi: S_n \rightarrow S_n$ and $\varphi(E_1) = E_2$.

Also define $f: G(E_1) \rightarrow G(E_2); \rho \mapsto \varphi \rho \varphi^{-1}$. We show that f is a bijection.

First of all we must verify that $f(\rho) \in G(E_2)$ for every ρ in $G(E_1)$. Since

$\rho \in G(E_1)$, there exists an integer m such that $\rho^m = e_1$, for some e_1 in

$I_k^n(E_1)$, by Corollary 6. Next we show that $[f(\rho)]^m = e_2$ for some e_2 in

$I_k^n(E_2)$. Let $j \in E_2$; then $[f(\rho)]^m(j) = (\varphi \rho \varphi^{-1})^m(j) = (\varphi \rho^m \varphi^{-1})(j) = (\varphi e_1 \varphi^{-1})(j)$.

But $\varphi^{-1}(j) \in E_1$, thus $e_1[\varphi^{-1}(j)] = \varphi^{-1}(j)$. Hence $[f(\rho)]^m(j) = \varphi[\varphi^{-1}(j)] = j$

for every $j \in E_2$. Also if $j \notin E_2$, then $[f(\rho)]^m(j) = (\varphi e_1 \varphi^{-1})(j) \in E_2$.

Therefore $[f(\rho)]^m \in I_k^n(E_2)$ and thus $f(\rho) \in G(E_2)$ for every $\rho \in G(E_1)$.

Now, in Lemma 1 we showed that a transformation like f is automatically

injective, therefore we must only verify that f is surjective. Let $s \in G(E_2)$

and then illustrate a $\rho \in G(E_1)$ such that $f(\rho) = s$. Since s is in $G(E_2)$,

there exists an integer m such that $s^m = e_2$ for some e_2 in $I_k^n(E_2)$. Define

ρ by: $\rho[\varphi^{-1}(i)] = \varphi^{-1}[s(i)]$ for every $i \in S_n$. Since φ^{-1} is injective,

$\rho \in T_n$. Furthermore $\varphi \rho \varphi^{-1}(i) = s(i)$ or $[f(\rho)](i) = s(i)$ for all i in S_n .

Therefore $f(\rho) = s$. Finally we can show $\rho \in G(E_1)$ by an argument parallel

to the one used above to show every image under f is in $G(E_2)$. Thus f is

surjective. Hence f is a bijection and therefore $|G(E_1)| = |G(E_2)|$. \blacktriangle

Example from T_3 : Let $E_1 = \{1, 2\}$ and $E_2 = \{1, 3\}$. Then $I_2^3(E_1) = \{4, 5\}$ and $I_2^3(E_2) = \{3, 9\}$. Also $G(E_1) = \{4, 11, 5, 10\}$ where 4, 11 gen 4 and 5, 10 gen 5, and $G(E_2) = \{3, 25, 9, 19\}$ where 3, 25 gen 3 and 9, 19 gen 9.

Example from T_5 : We give an example to illustrate that in general

$|G_e|$ need not be a constant for every e in $I_k^n(E)$, as did in fact occur in the last example.

Let $E = \{1, 2\}$, $e_1 = (12111)$ and $e_2 = (12112)$. Then $e_1, e_2 \in I_2^5(E)$, however $|G_{e_1}| = 16$, while $|G_{e_2}| = 10$.

Now consider I_1^n . There are n idempotents in I_1^n . If $e \in I_1^n$, then there exists an $i \in S_n$ such that $e = (ii \cdots i)$. We also note that $|G_e| = |G\{i\}|$.

Lemma 7: For $e \in I_1^n$, $|G_e|$ equals the number of rooted trees on n vertices.

Proof: Let $e = (ii \cdots i)$ and let $s \in G_e$. Since s gen e , there exists an integer m such that $s^m(h) = i$ for all $h \in S_n$. Therefore $i = s^m(s(i)) = s^{m+1}(i) = s(s^m(i)) = s(i)$. If $s^j(h) = i$, then $s^{j+k}(h) = i$ for every integer k . Hence, in order that $s^m(h) = i$, all that is necessary is that $s^j(h) = i$ for some $j \leq m$. Therefore the characterization for s is that every element of S_n must have a power equal to i . So, we now represent s by a rooted tree on n vertices, where each vertex represents an element of S_n and the adjacent vertex toward the root is the image under s of its predecessor. The root will be i , since $s(i) = i$. Now if every element of S_n has a power under s equal to i , its tree will be one of these types. Conversely, every tree of this type corresponds to a transformation, which generates e , and the Lemma follows. \blacktriangle

Lemma 8: \mathcal{A}_n has $n!$ generators.

Proof: \mathcal{A}_n is the only idempotent in the symmetric group on n elements. Therefore each of the $n!$ elements of the group must generate \mathcal{A}_n , by Lemma 6. If $s \in T_n$, but s is not in the symmetric group, then there exists a k in S_n such that $s(i) \neq k$ for all $i \in S_n$. Hence $s^j(k) \neq k$ for any j and thus $s^j \neq \mathcal{A}_n$ for any j . Therefore s cannot generate \mathcal{A}_n , so that \mathcal{A}_n has exactly $n!$ generators. \blacktriangle

Lemma 9: If $s \in G(E)$, then $s(E) = E$.

Proof: Assume that $|E| = k$. Since $s \in G(E)$, there exists an $e \in I_k^n(E)$ such s gen e . Thus there exists an integer m such that $s^m = e$ or such that

$s^m(i) = i$ for all $i \in E$. Now, if $i \in E$, then $s(i) = s(s^m(i)) = s^m(s(i)) = e(s(i))$. That is, e fixes $s(i)$ and hence $s(i) \in E$ for every $i \in E$.

Finally assume there exist $i, \bar{i} \in E$ such that $s(i) = s(\bar{i})$. Then

$$i = s^m(i) = s^{m-1}(s(i)) = s^{m-1}(s(\bar{i})) = s^m(\bar{i}) = \bar{i}. \text{ Therefore } |s(E)| = k$$

and hence $s(E) = E$. \blacktriangle

Lemma 10: Any idempotent in I_k^n has at least $k!$ generators.

Proof: Let $e \in I_k^n$. Then there exists an $E \subseteq S_n$ such that $|E| = k$

and $e(i) = i$ for all i in E . Without loss of generality take $E = \{1, 2, \dots, k\}$.

Suppose $e(i) = t_i$ for $i = k+1, \dots, n$. Then put $B \equiv \{\rho \in T_k : \rho(E) = E\}$. B

can be thought of as the symmetric group on k elements, so that $|B| = k!$

We will now extend each ρ in B to S_n in such a way that $\rho \in T_n$ and ρ gen e .

Note that for every $\rho \in B$ there exists an integer m such that $\rho^m = \frac{1}{k}$.

Hence $\rho^m(i) = i$ for $i = 1, \dots, k$. Now let ρ be arbitrary in B . Then

$\rho(i) = \alpha_i$ and $\rho^m(i) = i$ for $i = 1, \dots, k$. The problem is to find q_i such

that if $\rho(i) = q_i$, then $\rho^m(i) = t_i = e(i)$, for $i = k+1, \dots, n$. Since e

is an idempotent $t_i \in E$ and hence $\rho^m(t_i) = t_i$ for $i = k+1, \dots, n$. Thus in

order to extend ρ to S_n define $\rho(i) = \rho(t_i)$ for $i = k+1, \dots, n$. Then

$$\rho^m(i) = \rho^{m-1}(\rho(i)) = \rho^{m-1}(\rho(t_i)) = \rho^m(t_i) = t_i = e(i) \text{ for } i = k+1, \dots, n.$$

Hence $\rho^m(i) = e(i)$ for $i = 1, 2, \dots, n$ or $\rho^m = e$. Thus we have constructed

$k!$ generators for e . \blacktriangle

Remark: This is a sharp lower bound, however for $n \geq 5$ and $k \neq n$, it is not useful as an approximation for $|G_e|$.

Theorem 8: In T_n , if $|E| = k$, then $k! \cdot k^{n-k} \leq |G_e| \leq k! \cdot n^{n-k}$.

Proof: By Lemma 10, every idempotent of $I_k^n(E)$ has at least $k!$

generators and there are k^{n-k} idempotents in $I_k^n(E)$. Hence $|G(E)| \geq k! \cdot k^{n-k}$.

Now let $s \in G(E)$. By Lemma 9, $s(E) = E$, hence $s \in P_E = \{\rho \in T_n : \rho(E) = E\}$.

Therefore $G(E) \subseteq P_E$. But from Theorem 1, we know $|P_E| = k! \cdot n^{n-k}$. Thus

$$|G(E)| \leq k! \cdot n^{n-k}. \blacktriangle$$

Example: If we let $n = 3$ and $k = 2$, then $|G(E)| = 4$. The theorem predicts $2 \leq |G(E)| \leq 6$. However, if we let $n = 5$ and $k = 2$, then $|G(E)| = 92$, while the theorem predicts $16 \leq |G(E)| \leq 250$.

VI. CLASSES OF SUBSEMIGROUPS

We shall now construct various classes of subsemigroups, which assist in the counting of the subsemigroups of various orders. These classes also help us to enumerate the subsemigroups of a given order, which are τ -isomorphic (and thus also isomorphic under the usual concept of isomorphism).

Definition: Let $E \subseteq S_n$ such that $|E| = \ell$.

$$C_{k,\ell}^E \equiv \left\{ \begin{array}{l} \text{subsemigroups, } \sigma, \text{ of } T_n: |\sigma| = k \text{ and } \sigma \text{ contains} \\ \text{idempotents from } I_\ell^n(E), \text{ but no other idempotents} \\ \text{from } I_\ell^n. \end{array} \right\}$$

Special Case: If $\ell = 1$ and $E = \{q\}$, then denote the above collection by $C_{k,1}^q$ vice $C_{k,1}^{\{q\}}$.

Theorem 9: $|C_{k,\ell}^E|$ is independent of E and if $E, E' \subseteq S_n$ such that $|E| = |E'| = \ell$, then for every subsemigroup σ in $C_{k,\ell}^E$, there exists a subsemigroup σ' in $C_{k,\ell}^{E'}$ such that $\sigma \cong \sigma'$.

We will not prove this theorem here, as it is a special case of Theorem 10. However a better "feel" is obtained for the relationship between these classes by seeing the more particular case first.

Definition: Let $A_1, \dots, A_r \subseteq S_n$ such that $|A_j| = \ell$ for $j=1, \dots, r$.

$$C_{k,\ell}^{A_1, \dots, A_r} \equiv \left\{ \begin{array}{l} \text{subsemigroups, } \sigma, \text{ of } T_n: |\sigma| = k \text{ and } \sigma \text{ contains} \\ \text{idempotents from } I_\ell^n(A_j) \text{ for some } j \text{ (not necessarily} \\ \text{just one } j), \text{ but no other idempotents from } I_\ell^n. \end{array} \right\}$$

Note: k may vary from 1 to n^n , but $\ell \leq n$.

Example from T_3 : Let $E_1 = \{1,2\}$ and $E_2 = \{1,3\}$. Then $I_2^3 = \{3,4,5,9,15,24\}$, $I_2^3(E_1) = \{4,5\}$ and $I_2^3(E_2) = \{3,9\}$.

$C_{k,2}^{\bar{E}_1, \bar{E}_2}$ = the collection of all order k subsemigroups of T_3 ,
 which contain some combination of the idempotents
 $\{3,4,5,9\}$, but do not contain 15 or 24.

Theorem 10: Suppose $|A_j| = |B_j| = k$, for $j = 1, \dots, r$ and
 $|A_i \cap A_j| = |B_i \cap B_j|$ for all $i, j=1, \dots, r$. Then $|C_{k,l}^{A_1, \dots, A_r}| = |C_{k,l}^{B_1, \dots, B_r}|$
 and for every σ in $C_{k,l}^{A_1, \dots, A_r}$ there exists a σ' in $C_{k,l}^{B_1, \dots, B_r}$ such that
 $\sigma \cong \sigma'$.

Proof: The fact that $|A_j| = |B_j|$ and $|A_i \cap A_j| = |B_i \cap B_j|$, for all
 i and j yields a bijection $\varphi: S_n \rightarrow S_n$ such that $\varphi(A_j) = B_j$, $j=1, \dots, r$.
 For simplicity let $X \equiv C_{k,l}^{A_1, \dots, A_r}$ and $Y \equiv C_{k,l}^{B_1, \dots, B_r}$. Now define
 $f: X \rightarrow Y$; $\sigma \mapsto \sigma'$, where $\sigma' = \varphi \rho \varphi^{-1} : \rho \in \sigma$. We must first verify that $f(\sigma)$
 as defined is in Y for every σ in X . If $\rho_1, \rho_2 \in \sigma$ such that $\rho_1 \neq \rho_2$, then
 $\varphi \rho_1 \varphi^{-1} \neq \varphi \rho_2 \varphi^{-1}$. Thus $|\sigma'| = k$. Next we assume σ' contains an idempotent
 e from $I_{\mathcal{Q}}^n$ that is not in $I_{\mathcal{Q}}^n(B_j)$ for any j and obtain a contradiction.
 Since $e \in \sigma'$, there exists a $\rho \in \sigma$ such that $e = \varphi \rho \varphi^{-1}$ or $\rho = \varphi^{-1} e \varphi$. Also
 $e \in I_{\mathcal{Q}}^n$, therefore e fixes l elements, but by assumption e does not fix all
 of the elements of any B_j . Therefore there exist $i_1 \in B_1, \dots, i_r \in B_r$ such
 that $e(i_j) \neq i_j$ ($j=1, \dots, r$). Now since φ is injective $\rho[\varphi^{-1}(i_j)] =$
 $\varphi^{-1} e \varphi[\varphi^{-1}(i_j)] = \varphi^{-1}[e(i_j)] \neq \varphi^{-1}(i_j)$. We note that $\varphi^{-1}(i_j) \in A_j$ for
 every j and hence $\rho \notin I_{\mathcal{A}}^n(A_j)$ for any j . But $\rho \in I_{\mathcal{Q}}^n$, hence $\rho \notin \sigma$, a
 contradiction. Therefore the only idempotents from $I_{\mathcal{Q}}^n$ that σ' contains
 are from $I_{\mathcal{Q}}^n(B_j)$ for some j . And for any idempotent e in σ that is in
 $I_{\mathcal{A}}^n(A_j)$, $\varphi e \varphi^{-1} \in I_{\mathcal{Q}}^n(B_j)$, so that σ' does contain some idempotent from
 $I_{\mathcal{Q}}^n(B_j)$. Thus $\sigma' = f(\sigma)$ is in Y . Next we show that f is a bijection.
 Assume $\sigma_1, \sigma_2 \in X$ such that $\sigma_1 \neq \sigma_2$. Then there exists a $\rho_1 \in \sigma_1$ such that
 $\rho_1 \neq \rho$ for every $\rho \in \sigma_2$. Hence $\varphi \rho_1 \varphi^{-1} \neq \varphi \rho \varphi^{-1}$ for every $\rho \in \sigma_2$. There-
 fore σ'_1 has an element, namely $\varphi \rho_1 \varphi^{-1}$, which is not in σ'_2 . Thus $f(\sigma_1) \neq f(\sigma_2)$,

so that f is an injection. Now, let $\sigma' \in Y$, and define $\sigma = \left\{ \varphi^{-1} \rho \varphi : \rho \in \sigma' \right\}$. Then by arguments similar to those above we can show that $\sigma \in X$. Also $f(\sigma) = \left\{ \varphi(\varphi^{-1} \rho \varphi) \varphi^{-1} : \rho \in \sigma' \right\} = \sigma'$, therefore f is a surjection. Thus f is a bijection. Hence $|X| = |Y|$. Finally we show $\sigma \approx f(\sigma)$, by defining $\eta: \sigma \rightarrow f(\sigma); \rho \mapsto \varphi \rho \varphi^{-1}$. As above η is a surjection. Thus we apply Lemma 1 and conclude $\sigma \approx f(\sigma)$. Δ

Remark: From the definition we see that $C_{\ell_2, \ell_2}^{A, B} = C_{\ell_2, \ell_2}^{B, A}$. Thus although we may not have the hypothesis $|A_i \cap A_j| = |B_i \cap B_j|$ satisfied at first, we may be able to satisfy it by some rearrangement of the collection $\{B_1, \dots, B_r\}$.

Since every finite semigroup contains at least one idempotent, every subsemigroup of T_n must be in some C-class. Therefore the union of all these different types of classes would yield all the subsemigroups of T_n . However, they are not disjoint, in general, since a subsemigroup could contain idempotents from I_{ℓ}^n for several ℓ . We will prove one more theorem concerning these classes of subsemigroups and then show how to use them to obtain a decomposition of the collection of subsemigroups of T_n of a given order.

Notation: We will denote $C_{\ell_2, \ell_2}^{B_1, \dots, B_s}$ by $C_{\ell_2, \ell_2}^{\beta_\alpha}$. Thus β_α stands for a given collection of s subsets of S_n , each of order ℓ_2 . Then $\bigcup_{\alpha} C_{\ell_2, \ell_2}^{\beta_\alpha}$ is a union of C-classes and includes the class $C_{\ell_2, \ell_2}^{B_1, \dots, B_s}$ for every possible collection of subsets $\{B_1, \dots, B_s\}$, where $|B_j| = \ell_2, j=1, \dots, s$.

Theorem 11: Let $|A_i| = |A_i'| = \ell_1$ and $|A_i \cap A_j| = |A_i' \cap A_j'|$ for $i, j=1, \dots, r$. Let $X \equiv C_{\ell_2, \ell_2}^{A_1, \dots, A_r} \cap \left(\bigcup_{\alpha} C_{\ell_2, \ell_2}^{\beta_\alpha} \right)$ and $Y \equiv C_{\ell_2, \ell_2}^{A_1', \dots, A_r'} \cap \left(\bigcup_{\alpha} C_{\ell_2, \ell_2}^{\beta_\alpha} \right)$. Then $|X| = |Y|$ for every subsemigroup σ in X there exists a σ' in Y such that $\sigma \approx \sigma'$.

Proof: First of all construct a bijection $\varphi: S_n \rightarrow S_n$ such that $\varphi(A_i) = A_i'$ for all i . Then define $f: X \rightarrow Y; \sigma \mapsto \sigma'$, where $\sigma' = \{\varphi \alpha \varphi^{-1} : \alpha \in \sigma\}$. Let $\sigma \in X$; we show $\sigma' = f(\sigma) \in Y$. First of all since $\sigma \in C_{k_1, \dots, k_r}^{A_1, \dots, A_r}$, Theorem 10 implies $\sigma' \in C_{k_1', \dots, k_r'}^{A_1', \dots, A_r'}$. Also $\sigma \in \bigcup_{\alpha} C_{k_1, k_2}^{B_{\alpha}}$ implies there exists an α^* such that $\sigma \in C_{k_1, k_2}^{B_{\alpha^*}} = C_{k_1, k_2}^{B_1, \dots, B_s}$. Again by application of Theorem 10, $\sigma' \in C_{k_1, k_2}^{\varphi(B_1), \dots, \varphi(B_s)} \subseteq \bigcup_{\alpha} C_{k_1, k_2}^{B_{\alpha}}$. Hence σ' is in Y . As before f is injective. Let σ' be in Y and define $\sigma \equiv \{\varphi^{-1} \alpha \varphi : \alpha \in \sigma'\}$. Then $\sigma' = f(\sigma)$ and by a double application of Theorem 10, similar to that done above, we can show $\sigma \in X$. Hence f is a surjection. Thus we conclude that f is a bijection and $|X| = |Y|$. From Theorem 10 we also conclude $\sigma \cong f(\sigma)$. Δ

Let us now illustrate how this last theorem allows us to decompose the subsemigroups of T_3 of order 3. By digital computation we have discovered that there are 86 order 3 subsemigroups of T_3 ⁶. By checking our list of subsemigroups, we count 17 in C_3^1 . Theorem 10 implies there are also 17 in C_3^2 and C_3^3 . This accounts for 51 subsemigroups. We remove these 51 from the collection and consider those remaining. Of these, 7 are in $C_{3,1}^{\{1\}, \{2\}}$. According to Theorem 11 (take $B_{\alpha_1} = \{1\}$, $B_{\alpha_2} = \{2\}$ and $B_{\alpha_3} = \{3\}$), there are also 7 in $C_{3,1}^{\{1\}, \{3\}}$ and $C_{3,1}^{\{2\}, \{3\}}$ ^{α_2} . We also remove these. This leaves 14 subsemigroups. Three of these are in $C_{3,2}^{\{1,2\}}$ and thus by Theorem 11, there are three in each of $C_{3,2}^{\{1,3\}}$ and $C_{3,2}^{\{2,3\}}$. Then, of the remaining five; one is in $C_{3,2}^{\{1,2\}, \{1,3\}}$ and again according to Theorem 11 one is also in each of $C_{3,2}^{\{1,2\}, \{2,3\}}$ and $C_{3,2}^{\{1,3\}, \{2,3\}}$. Finally there is one in each of the classes $C_{3,1}^{\{1\}, \{2\}, \{3\}}$ and $C_{3,3}^{\{1,2,3\}}$. This accounts for all 86

⁶See Appendix A

subsemigroups of order 3. Using these numbers, we at once conclude that the maximum number of non- τ -isomorphic subsemigroups of order 3 is $17 + 7 + 3 + 1 + 1 + 1 = 30$. Then using Lemma 1, we find that several of these 30 are τ -isomorphic and that the actual number of subsemigroups of order 3 that are distinct up to a τ -isomorphism is 16.

APPENDIX A

NUMBER OF SUBSEMIGROUPS OF T_3

Order of Subsemigroup	Number of Subsemigroups
1	10
2	45
3	86
4	136
5	192
6	206
7	186
8	144
9	109
10	63
19	0
20	0
21	1
22	1
23	3
24	1
25	0
26	0
27	1

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13. ABSTRACT			

Both necessary conditions and sufficient conditions in order that a subset of a finite transformation semigroup be a subsemigroup are developed in this paper. The existence of several subsemigroups of various orders is established. Also, some results concerning idempotents and generators of idempotents are proved. Then certain classes of subsemigroups are defined according to their idempotent structure and isomorphisms are demonstrated between these classes. Examples from the transformation semigroup on three elements are supplied throughout the paper.

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Transformation semigroup						
Idempotent						
Subsemigroup						
Isomorphism						

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